SOLUTIONS TO EXAM 2, MATH 10550

1. *Find the critical numbers of $f(x) = \dfrac{1}{6}x^2 - x^{1/3}$.*

Solution:

We have
$$f'(x) = \frac{1}{3}x - \frac{1}{3}x^{-2/3} = \frac{1}{3}\left[\frac{x^{5/3} - 1}{x^{2/3}}\right].$$
So $x = 0$ is a critical point. For $x \neq 0$, the equation $f'(x) = 0$ can be written
$$x^{5/3} = 1, \text{ that is, } x = 1.$$
Hence the critical numbers of f are $x = 0$, $x = 1$.

2. *Given $f'(x) = \frac{x^2 - 1}{(x^2+1)^3}$, find the correct statement.*

f has a local maximum at $x = -1$.
f has a local maximum at $x = 1$ and a local minimum at $x = -1$.
f has no local maximum.
f has no local minimum.
f has a local minimum at both $x = -1$ and $x = 1$.

Solution:

The critical points of f occur where $f'(x) = 0$, they are $x = -1$, $x = 1$. Since $f'(x)$ is positive in the intervals $(-\infty, -1)$ and $(1, +\infty)$ and negative in the interval $(-1, 1)$, we conclude that f has a local maximum at $x = -1$ and f has a local minimum at $x = 1$. Only the first statement above is true.

3. *Evaluate the following limit*
$$\lim_{x \to -\infty} \frac{\sqrt{9x^2 + 7x^{1/3}}}{8x + 7}.$$

Solution: Note that since $\sqrt{x^2} = -x$ if $x < 0$, we get

$$\begin{aligned}
\lim_{x \to -\infty} \frac{\sqrt{9x^2 + 7x^{1/3}}}{8x + 7} &= \lim_{x \to -\infty} \frac{(\sqrt{9x^2 + 7x^{1/3}})/x}{(8x + 7)/x} \\
&= \lim_{x \to -\infty} \frac{-\sqrt{(9x^2 + 7x^{1/3})/x^2}}{8 + (7/x)} \\
&= \lim_{x \to -\infty} \frac{-\sqrt{9 + (7/x^{5/3})}}{8 + (7/x)} \\
&= \frac{-\sqrt{9}}{8} \\
&= \frac{-3}{8}.
\end{aligned}$$

4. *Evaluate*
$$\lim_{x \to \infty} \left(\sqrt{x^2 + x} - x \right).$$

Solution:

$$
\begin{aligned}
\lim_{x \to \infty} \left(\sqrt{x^2 + x} - x \right) &= \lim_{x \to \infty} \left(\sqrt{x^2 + x} - x \right) \frac{\sqrt{x^2 + x} + x}{\sqrt{x^2 + x} + x} \\
&= \lim_{x \to \infty} \frac{x^2 + x - x^2}{\sqrt{x^2 + x} + x} \\
&= \lim_{x \to \infty} \frac{x}{\sqrt{x^2 + x} + x} \\
&= \lim_{x \to \infty} \frac{x/x}{(\sqrt{x^2 + x} + x)/x} \\
&= \lim_{1 \to \infty} \frac{x}{\sqrt{(x^2 + x)/x^2} + 1} \\
&= \lim_{x \to \infty} \frac{1}{\sqrt{1 + 1/x} + 1} \\
&= \frac{1}{\sqrt{1} + 1} \\
&= \frac{1}{2}
\end{aligned}
$$

5. *Find the number of inflection points of* $f(x) = x^4 + 4x^3 + 12x^2$.
Solution: $f'(x) = 4x^3 + 12x^2 + 24x$, and $f''(x) = 12x^2 + 24x + 24$. The inflection point occur where f'' switches sign. Since f'' is continuous here, this must happen where $f''(x) = 0$, namely, $12x^2 + 24x + 24 = 0$,
or $x^2 + 2x + 2 = (x + 1)^2 + 1 = 0$.
There is no root for the equation. So the number of inflection points is 0.

6. *Consider the function* $f(x) = x + \frac{1}{x}$, *defined for* $0 < x \le 10$. *Find the correct statement.*

(a) f has precisely one critical point.
(b) The graph of f is concave down (on the entire interval).
(c) f has no maximum and f is decreasing (on the entire interval) .
(d) f has no minimum and f is decreasing (on the entire interval).
(e) f has precisely one inflection point.

Solution: (a) $f'(x) = 1 - \frac{1}{x^2} = 0 \Rightarrow x = 1 \in (0, 10)$, so f has precisely one critical point.
(b) $f''(x) = \frac{2}{x^3} > 0$ for $0 < x \le 10$. So the graph of f is concave up.
(c)/(d) f is decreasing on (0,1) and increasing on (1, 10). f has no maximum and a minimum on $x = 1$.
(e) $f''(x) = \frac{2}{x^3} > 0$. So f has no inflection point.

7. *Suppose $f(x)$ is continuous on $[0,1]$ and differentiable on $(0,1)$. If $f(0) = 0$ and $0 \leq f'(x) \leq \frac{1}{2}$, what is the largest $f(x)$ can be when $x = 1$?*
   Solution:
Since f is continuous on $[0,1]$ and differentiable on $(0,1)$, f satisfies the hypotheses of the Mean Value Theorem. By the Mean Value Theorem, there is a real number c in the interval $(0,1)$ such that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = f(1) - f(0) = f(1) - 0 = f(1)$$

We're given that $f'(x) \leq \frac{1}{2}$ for all real numbers in the interval $(0,1)$, so in particular, $f'(c) \leq \frac{1}{2}$. Thus, $f(1) = f'(c) \leq \frac{1}{2}$, and the largest $f(1)$ can be is $\frac{1}{2}$.

8. *Find the linearization $L(x)$ of the function $f(x) = (x+1)\cos x$ at 0 (or $a = 0$).*
   Solution:
First we must find the slope of the tangent line at $x = 0$. We compute

$$f'(x) = (x+1)(-\sin x) + \cos x(1) = \cos x - x\sin x - \sin x.$$

At $x = 0$, we have $f'(0) = \cos 0 - (0)\sin 0 - \sin 0 = 1 - 0 - 0 = 1$. Since $f(0) = (0+1)\cos 0 = 1$,

$$L(x) = f(0) + f'(0)(x - 0) = 1 + x.$$

9. *Use a linear approximation to estimate $\sin\frac{\pi}{50}$.*
   Solution: Recall that the linear approximation equation is $T(x) = f(a) + f'(a)(x-a)$, where a is a point that is the point closest to $\frac{\pi}{50}$ such that $sin(a)$ can be easily computed. It is easy to see that 0 is the best choice for a. Now $f(0) = sin(0) = 0$. and $f'(x) = cos(x)$, so $f'(0) = 1$. So $T(x) = 0 + 1(x - 0) = x$. Therefore, $T(\frac{\pi}{50}) = \frac{\pi}{50}$ is the linear approximation.

10. *Let*
$$f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + 10.$$
*On which of the intervals given below is the graph of f **both** increasing and concave upward (on the entire interval)?*

   Solution: This is equivalent to finding an interval where both $f'(x) > 0$ and $f''(x) > 0$ for all x in that interval Note that $f'(x) = x^2 - 3x + 2 = (x-1)(x-2)$ Notice that $f'(x) > 0$ when $1 < x$ and $x > 2$. Next, $f''(x) = 2x - 3$, so $f''(x) > 0$ when $x > \frac{3}{2}$. Taking the intersection of the intervals for which both $f'(x) > 0$ and $f''(x) > 0$ yields $(2, \inf)$

11. *Show that there is precisely one (real) root of the equation*

$$x^5 + 4x - 3 = 0.$$

Justify your answer and identify the theorem(s) you use.

Solution:

Let $f(x) = x^5 + 4x - 3$. Notice f is continuous and differentiable everywhere, so the hypothesis of the intermediate value and mean value theorems are satisfied. Now, since

$$f(0) = -3 < 0, \quad \text{and} \quad f(1) = 2 > 0,$$

the function $f(x)$ must attain the value zero at some point c in the interval $[0, 1]$, by the intermediate value theorem.

If $f(x)$ had two distinct zeroes c and d, then there is a number e between c and d for which $f'(e) = 0$, by the mean value theorem.

But

$$f'(x) = 5x^4 + 4$$

is always positive, so no such e exists and we cannot have two distinct roots c and d .

12. *An eccentric mathematician enjoys inflating and releasing cube-shaped balloons. Assume that gas is being pumped into the balloon at the rate of three cubic feet per second. When the edge is one foot long, what is the rate of change of the surface area of the balloon?*

Solution: Let x be the length of each side of the cube. Then we have the formulas for volume: $V = x^3$ and surface area: $S = 6x^2$. We are given $\frac{dV}{dt} = 3$ and we want to find $\frac{dS}{dt}$ when $t = 1$.

Start by using implicit differentiation to find $\frac{dV}{dt} = 3$ and $\frac{dS}{dt}$.

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

and

$$\frac{dS}{dt} = 12x \frac{dx}{dt}.$$

We need to know what $\frac{dx}{dt}$ is when $x = 1$. Plugging what we are given into the equation for $\frac{dV}{dt}$ gives us $3 = 3(1^2)\frac{dx}{dt}$. Therefore, $\frac{dx}{dt} = 1$.

Now plug this into the equation for $\frac{dS}{dt} = 12(1)(1) = 12$. Therefore, when the side of the length is one foot long, the rate of change of the surface area is 12 feet squared per second.

13. *Consider the function $f(x) = 3x^4 - 4x^3 + 2012$ defined on $(-\infty, \infty)$.*
(1) Find the intervals on which f is increasing and decreasing.
(2) Find the intervals on which f is concave up and concave down and the points of inflection of f.
(3) Give the x and y values for the global/absolute minimum of f on the interval $(-\infty, \infty)$. Justify your answer.

Solution: (1)Since $f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1)$, f is increasing when $x > 1$ ($f' > 0$) and decreasing when $x < 1$ ($f' < 0$).

(2) Since $f''(x) = 36x^2 - 24x = 12x(3x - 2)$, f is concave up when $x < 0$ or $x > \frac{2}{3}$ ($f'' > 0$) and concave down when $0 < x < \frac{2}{3}$ ($f'' < 0$). The points of inflection of f are $x = 0, \frac{2}{3}$.

(3) By the first derivative test $f(x)$ has a local minimum at $x = 1$. This must be a global minimum for f Since the function is increasing on the interval $[1, \infty)$ and decreasing on the interval $(-\infty, 1]$.

The x and y values on the graph at $x = 1$ are given by $(1, 2011)$.